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Solution of Slightly Underexpanded Two-Dimensional and Axisymmetric Coflowing Jets

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A new approach is presented for solving the problem of a slightly underexpanded axisymmetric supersonic jet in a subsonic coflowing stream. A finite-difference technique is used to predict the complete jet plume flowfield including viscous mixing and jet entrainment effects. The problem is rendered parabolic through unique use of approximate inviscid intrinsic coordinates to estimate only the streamline curvatures. The general inviscid problem is formulated in terms of linearized perturbation potentials with solutions presented for both two-dimensional and axisymmetric flows. The underexpanded jet capability is demonstrated through comparison with other analytical models and experimental data.

Nomenclature

a_n	= zeros of Bessel function of order one
C	= constant in the eddy viscosity
C_p	= pressure coefficient
D	= jet diameter
h	= transverse distance of jet interface
h_b	= transverse distance of boattail
h_j	= initial jet height
h_s, h_n	= metric coefficients
H	= normalized transverse distance of jet interface
H_0	= total enthalpy
J_0	= Bessel function of order zero
J_1	= Bessel function of order one
m	= jet parameter defined by Eq. (8b)
M	= Mach number
n	= integer, or normal coordinate
p	= static pressure
P_r	= Prandtl number
$\mathcal{P}$	= function to account for nonuniform exit conditions
r	= transverse distance
r_j	= initial jet radius
R	= streamline curvature
S	= source strength per unit length
s	= longitudinal distance
T	= temperature
u	= perturbation velocity in longitudinal direction
U	= total velocity in longitudinal direction
v	= perturbation velocity in transverse direction
x	= longitudinal distance
x_c	= extent of potential core region
β	= Mach number compressibility parameters
γ	= ratio of specific heats
δ	= characteristic length
μ	= viscosity
ξ	= dummy variable representing longitudinal distance
ρ	= density
ϕ	= perturbation potential
Δu	= $u_{\mathrm{C.L.}} - u_{\min}$

Subscripts

0	= total or exit value
b	= boattail value
j	= jet value
t	= stagnation or turbulent value
r	= reference value
∞	= freestream value

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Introduction

THE problem addressed here was that of predicting the detail flow structure in the jet plume that occurs when an underexpanded hot jet exhausts into a cold coflowing subsonic stream (see Fig. 1). The ability to predict the details of the viscous mixing (and thus cooling) that takes place at the jet interface is critical to determining the thermodynamic signature of the vehicle producing the jet. It is therefore important to develop a reliable and efficient scheme for addressing this class of problems.

The goal of the current work was to develop an efficient finite-difference technique for addressing the case of axisymmetric, slightly underexpanded, hot jet flow in a subsonic mainstream. The approach used was a direct extension of a parabolic marching technique recently developed by Anderson[1] for internal axisymmetric flows at high Reynolds numbers. In that approach Anderson introduced the concept of using an approximate intrinsic coordinate system to formally produce a parabolic composite set of equations valid throughout the inviscid and viscous regions of the flow. An implicit finite-difference algorithm based on Keller's box scheme[2] was employed to obtain fast, accurate solutions, and numerous comparisons with experimental data have been achieved.[1,3,4]

In the current paper, this concept was extended to the coflowing jet case where the far-field condition becomes that of uniform flow. To do this, the work first concentrated on developing the approximate intrinsic coordinate system. To this end, the inviscid flow was first predicted using inviscid perturbation theory to effect a rather simple construction of the plume shape. The accuracy of this inviscid solution was assessed through detailed comparison with more complicated solution schemes that employ a method of characteristics representation of the jet flow. Thereafter, solution of the axisymmetric viscous plume problem was achieved for hot and cold subsonic and supersonic jets exhausting into either still air or a subsonic stream. The turbulence model used has been calibrated against experimental data for perfectly expanded subsonic and supersonic jets in earlier studies.[5,6] Underexpanded hot supersonic jets ($M_j = 2$) were computed

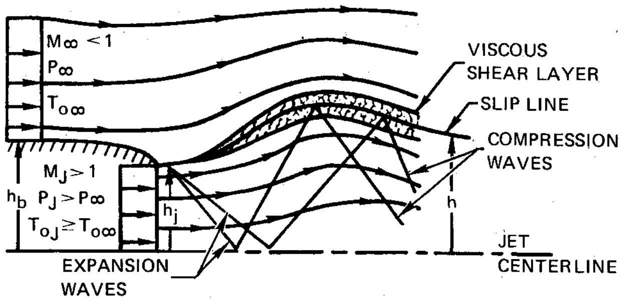


Fig. 1 Structure of an underexpanded jet.

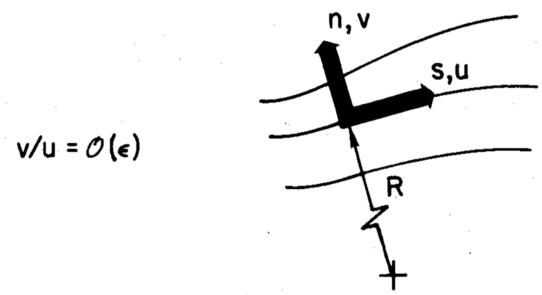


Fig. 2 Governing equation approximations.

for exit jet pressure ratios (p_j/p_∞) up to approximately 1.4 for flow into still air or a cold coflowing subsonic stream $(M_\infty = 0.8)$. Good comparisons were obtained with the experimental data of Seiner and Norum[7] for the case of an underexpanded supersonic freejet, and with the experimental data of Robinson et al.[8] for the case of a hot supersonic underexpanded jet coflowing in a subsonic stream.

Governing Equations—Viscous Flow

A. General Concepts

The overall approach taken here for representing viscous flow effects on the plume mixing characteristics follows directly from the work of Anderson.[1,3,4] For high Reynolds number internal flow in axisymmetric ducts, Anderson developed a novel parabolic marching technique based on the use of approximate intrinsic coordinates. The resulting governing equations represent a composite set valid in both the inviscid regions as well as the thin boundary-layerlike regions. Anderson found that through careful consideration of the streamline curvature term appearing in the normal momentum equation, he could reduce the problem to an initial value problem. As discussed in Ref. 1, solution of these equations produces the class of weak interaction solutions wherein viscous effects play a formally second-order but quantitatively significant role in establishing the details of the flow structure. The utility of this approach has been verified through extensive application to subsonic internal flow problems (see Ref. 4), and the present effort represents its application to external mixed flows with large regions of both subsonic and supersonic flow. A similar approach has been used by Edelman and Weilerstein[9] for supersonic mainstreams and the current work represents an extension of this earlier work to subsonic mainstreams.

B. Viscous Equations

The governing equations are recovered by first writing the full Navier-Stokes equations in a general orthogonal coordinate system for axisymmetric and two-dimensional flows and then particularizing them to "near-streamline" coordinates. As shown in Fig. 2, Prandtl-type approximations are then applied to this set in two principal ways: 1) it is assumed that convection normal to these coordinate lines is small; and 2) the dominant diffusion effect acts normal to the longitudinal coordinate direction. Both of these approximations are assumed to hold throughout the entire flowfield and thus are inherently tied to the use of an intrinsic coordinate system. The resulting equation set, written in terms of a streamline coordinate s and its orthogonal coordinate n, are given as:

$$\frac{\partial}{\partial s}(rh_n\rho u) + \frac{\partial}{\partial n}(rh_s\rho v) = 0 \tag{1}$$

$$\frac{\partial}{\partial s}(h_n r\rho u^2) + \frac{\partial}{\partial n}(h_s r\rho uv) + rh_n\frac{\partial p}{\partial s} = \frac{\partial}{\partial n}\left[\frac{h_s r}{h_n}\mu_T\frac{\partial u}{\partial n}\right] \tag{2}$$

$$\frac{1}{h_n}\frac{\partial p}{\partial n} - \frac{\rho u^2}{R} = 0 \tag{3}$$

$$\frac{\partial}{\partial s}(h_n r\rho u H_0) + \frac{\partial}{\partial n}(h_s r\rho v H_0) = \frac{\partial}{\partial n}\left\{\frac{h_s r}{h_n}\frac{\mu_T}{P_{r_T}}\left[\frac{\partial H_0}{\partial n} + (P_{r_T} - 1)u\frac{\partial u}{\partial n}\right]\right\} \tag{4}$$

where h_s and h_n are the coordinate scale factors and R the radius of curvature of the streamlines.

Note that solution of these equations requires prior knowledge of the streamline shapes and, in particular, the streamline curvature for use in the normal momentum equation. Anderson[1] has shown that equations of this type are formally parabolic throughout the entire flowfield and he gives strong arguments for solving them as an initial value problem. Thus, all flow properties, including the pressure level, can be recovered in a marching solution technique.

For the case of coaxial jet flows, the boundary and initial conditions employed differ slightly from those used in previous works.[1,3,4] The initial station profiles of longitudinal velocity, pressure, and temperature are assumed known, while in the far field, the applied conditions are $u = U_\infty$, $p = p_\infty$, and $T = T_\infty$, while at the jet centerline, $\partial u/\partial r = 0$ and $\partial T/\partial r = 0$.

C. Turbulence Model

A simple eddy viscosity model formulated in Ref. 5 has been used in this study for jet flows. This model was found through numerical experiments[5,6] to provide a reasonable representation of the turbulence effects for both free and coflowing jet flows over a wide range of test conditions. Thus, it is employed in this study to allow a realistic assessment of the overall jet calculation concept and represents one area where future studies might employ a more comprehensive model, such as one based on a differential equation set if experimental comparisons indicated such a need.

In the present model, the turbulent viscosity term is given everywhere by a relationship of the form

$$\mu_T = C\rho\delta(u_{max} - u_{min}) \tag{5}$$

where ρ is the density, δ a characteristic length scale, $(u_{max} - u_{min})$ a measure of turbulent kinetic energy, and C an empirical constant.

The mixing process is assumed to take place in three principal regions, each with their own version of the constant to be used in Eq. (5). The three regions are (I) the initial mixing region (potential core region), (III) a fully turbulent region, and (II) an intermediate or transitional region.

The initial mixing region (region I) has been documented by Schlichting,[10] where analytical solution of this problem has been given in similarity variables. Here δ is taken as the distance between the two points in the shear layer where $[(u - u_{min})/(u_{max} - u_{min})]^2$ varies from 0.1 to 0.9. By matching the measured width of the mixing zone, a value of 0.014 was obtained for the constant C.

For the fully turbulent region (region III), Hinze[11] has taken δ as the jet half width and assigned C a value of 0.0256, on the basis of experimental data.

For the intermediate or transitional region, an approach similar to that of Chen[12] is employed, where it is assumed that the flow becomes fully turbulent at $x=2x_c$, where x_c represents the end of potential core region. It was assumed that the potential core ends when the centerline velocity of the jet differs by more than 1% from the jet exit velocity. With this, the eddy viscosity model used here can be summarized as follows:

$$\mu_T = \begin{cases} \mu_{T_I} & (x \le x_c) \\ (2-x/x_c)\mu_{T_I} + (x/x_c - 1)\mu_{T_{III}} & (x_c \le x \le 2x_c) \\ \mu_{T_{III}} & (x \ge 2x_c) \end{cases} \tag{6}$$

D. Numerical Method

The set of equations (1-6) forms a set of quasilinear, first-order partial-differential equations. Anderson[1] has shown that with the specification of the curvature, these equations become parabolic, and as such can be solved with a forward marching numerical method once the relationship between μ and the mean flow is specified. This is a critical point in the current approach, since it allows one to calculate the entire flowfield including the pressure field in a forward marching scheme, even in the large subsonic inviscid regions. This is possible because the primary "elliptic" property of these solutions is communicated through the influence of streamline curvature. That is, the flow at any given streamwise station "knows" that the flow is going to turn farther downstream because of the assumed streamline curvatures. This information is contained in the choice of coordinate systems and is reflected principally in the normal momentum equation (3) which states that the transverse static pressure gradient in the direction of the principal curvature R is determined by the local velocity for the viscous flow and the radius of curvature R of the flow streamlines. This would be an exact equation for inviscid flow in intrinsic coordinates if R were the actual streamline curvature. Thus, one important feature of the current approach is its replacement of the actual streamline curvature with an approximate flow streamline curvature, and the quantitative value of the approach is critically tied to the ability to determine good approximations to the streamline curvatures. This point is covered in detail in the next section and it remains here to set the means of solving the governing equations for a given orthogonal coordinate system and streamline curvatures.

Solutions were obtained here using a finite-difference representation of the quasilinearized stream function form of the differential equations that employed the two-point, centered-difference scheme of Keller.[2] For the full set of governing equations the resulting matrix equations are block tridiagonal which are solved by a straightforward block factorization. The curvature R [see Eq. (3)] of the streamlines was calculated at every point from the inviscid plume solutions as presented in the next section. Because only slight pluming, and thus small streamline turnings were anticipated in the regime of interest here, the intrinsic coordinates were approximated by cylindrical coordinates, so that the coordinate scale factors h_s and h_n of Eqs. (1-4) were unity. To obtain accurate representation of the imbedded viscous shear layer near the nozzle lip, a variable grid spacing was used (see Ref. 1). Also it was found that the supersonic portion of the flow could not be accurately represented unless the grid aspect ratio closely honored a CFL condition, here defined in terms of the jet exit Mach number. Typical computing time for solving the viscous equations on a Univac 1100 computer for a 50×50 finite-difference mesh was approximately 3 min including 1.5 min of compilation time.

Governing Equations—Inviscid Plumes

As pointed out previously, in order to achieve a solution of the viscous flow equations, an approximate representation of the streamlines of the flow is required to define the coordinate system and, subsequently, the curvature term appearing in the normal momentum equation, Eq. (3). For the class of slightly underexpanded jet flows to be considered here, it is anticipated that the streamlines will be only slightly displaced from a uniform flow state by both pluming and mixing effects. Thus for present purposes it is reasonable to use a simple Cartesian or cylindrical coordinate system and the inviscid streamline curvature as predicted by a small perturbation approach. Therefore, it is assumed that $h_s = h_n = 1$ and R is calculated from the linearized inviscid solution.

The inviscid problem of a slightly underexpanded supersonic jet in a subsonic coflowing stream is formulated here in terms of linearized perturbation potential equations in each of the supersonic and subsonic regions. This somewhat classic approach has been used by Pai[13] for the problem of a uniform parallel supersonic jet exhausting into uniform subsonic flow over a straight body of the same thickness as the jet exit. The basic equations for this case can be found in Ref. 13. Pai constructed solutions which are valid far downstream of the jet exit using a separation of variables approach to produce nearly periodic fundamental solutions. Klunker and Harder,[14] still using linear theory concepts, presented a more general formulation for the two-dimensional problem that is valid at the jet exit as well. The authors are unaware of any solutions given to date for these jet flow equations. The present approach presents a generalization of Klunker and Harder's[14] work to account for upstream boattail effects, nonparallel jet exit conditions, and axial symmetry. In addition, a solution technique is presented and demonstrated for several cases of practical interest.

In the present approach, the supersonic portion of the flow is solved using Laplace transform methods with the resulting solutions presented in terms of wave reflections and intersections for the two-dimensional case and in terms of Bessel functions for the axisymmetric case. The subsonic portion of the flow is solved using source distribution along the jet boundary for the two-dimensional case (as suggested by Klunker and Harder[14]), and using source distributions along the jet axis for the axisymmetric case. The final solution is achieved by satisfying the constraint that the slope and static pressure of the two flows match exactly at the jet interface (slip line or slip surface). This leads to a single integral equation for the slope h of jet interface. A novel feature of this approach is that the subsonic and supersonic flow regimes are coupled through a single equation for the jet interface. Once this is solved, the entire flowfield can then be written in terms of the shape (and derivatives) of the jet interface.

For the two-dimensional case, the equation for the slope of the jet interface in normalized form is found to be

$$H'(x) + \sum_{n=1}^{\infty} 2H'(x-2n\beta_r) - \frac{m}{\pi}\int_0^{\infty}\frac{H'(\xi)}{x-\xi}\mathrm{d}\xi$$
$$= 1 + \mathcal{P} + \frac{m}{\pi\beta_r C_{p_r}}\int_{-\infty}^{\infty}\frac{h_b'(\xi)}{x-\xi}\mathrm{d}\xi \tag{7}$$

where $H'(x)$ is a normalized version of the interface shape given by

$$H' \equiv h'/\beta_r C_{p_r} \tag{8a}$$

and

$$m \equiv \frac{p_\infty M_\infty^2}{p_j M_r^2}\frac{\beta_r}{\beta_\infty} \tag{8b}$$

$$\beta_\infty \equiv \sqrt{1-M_\infty^2} \qquad \beta_r \equiv \sqrt{M_r^2 - 1} \tag{8c}$$

with

$$C_{p_r} \equiv (2 - p_r/p_j - p_\infty/p_r)/\gamma M_r^2 \tag{8d}$$

$$M_r^2 \equiv \left(\frac{2}{\gamma-1} + M_j^2\right)\left(\frac{p_j}{p_r}\right)^{\frac{\gamma-1}{\gamma}} - \frac{2}{\gamma-1} \tag{8e}$$

The function h_b represents the boattail shape depicted in Fig. 1, while the function $\mathcal{P}$ is introduced to account for nonuniform flow conditions at the jet exit ($x=0$, $0 \leq h \leq 1$) as might be introduced by a divergent jet nozzle. For uniform parallel jet exit flow it is identically zero and its form for a polynomial representation of the jet exit conditions can be found in Ref. 6. For all cases, the jet conditions p_j and M_j are taken as their respective values at $x=0$, $h=1$. All the length terms presented in this discussion have been normalized with the jet height at the exit plane.

The reference pressure p_r is chosen for convenience only, as it represents the flow state about which the jet flow is linearized. For algebraic simplicity, one can choose $p_r = p_j$, whereas previous investigators[13,15] have found that linear theory can be applied to give good quantitative predictions over a wide range of jet pressures by taking $p_r = p_\infty$ (i.e., linearizing the jet flow around its isentropic plume state).

Note that for the case of uniform jet flow and no boattail effects, the function $H'(x)$ becomes independent of the jet strength p_j/p_∞, if one takes $p_r = p_j$. This approach then allows a single solution of Eq. (7) to apply for all jet pressure ratios through the transformation of Eq. (8a). However, this approach apparently has a limited jet pressure and Mach number range of application. It was found by most earlier investigators[13,15] that for free-jets, the better approach employed $p_r = p_\infty$.

In either case, the equation for H' can be solved numerically using a straightforward underrelaxation scheme to iteratively improve the integral term. Results for this case are briefly discussed in the next section. Once the shape function h' has been determined, the details of the entire flowfield can be constructed for the supersonic region ($h < h_j$) or subsonic region ($h > h_j$). For example, the longitudinal velocity components are given for $h < h_j$

$$\phi_x = u = \sum_{n=0}^{\infty} \frac{1}{\beta_r} h'[x - \beta_r(2n+1-h)] + \sum_{n=0}^{\infty} \frac{1}{\beta_r} h'[x - \beta_r(2n+1+h)] + \mathcal{P}_u \tag{9a}$$

while for $h > h_j$

$$\phi_x = u = \frac{1}{\pi\beta_\infty}\int_0^\infty \frac{h'(\xi)(x-\xi)}{[(x-\xi)^2 + \beta_\infty^2(h-1)^2]}\,\mathrm{d}\xi \tag{9b}$$

where $\mathcal{P}_u$, as does $\mathcal{P}$, accounts for nonuniform jet exit conditions as discussed in Ref. 6.

Once the velocity fields are known, the streamline curvature is given according to the relation

$$\frac{1}{R} \simeq \frac{\partial^2\phi}{\partial x \partial y} \tag{9c}$$

which is evaluated numerically using a simple central-difference representation of the velocity field provided in Eqs. (9).

For the axisymmetric case, the relationship for the interface shape is

$$\int_0^x 2H_a'(\xi)\,\mathrm{d}\xi + \sum_{n=1}^{\infty}\int_0^x 2H_a'(\xi)\cos[a_n(x-\xi)/\beta_r]\,\mathrm{d}\xi - m_a\int_0^\infty \frac{(x-\xi)S(\xi)}{[(x-\xi)^2+\beta_\infty^2]^{3/2}}\,\mathrm{d}\xi = 1 + \mathcal{P}_a + \frac{m_a}{\beta_r^2 C_{p_r}}\int_{-\infty}^0 \frac{(x-\xi)S_b(\xi)}{[(x-\xi)^2+\beta_\infty^2]^{3/2}}\,\mathrm{d}\xi \tag{10}$$

with

$$H_a'(x) = \int_0^\infty \frac{S(\xi)}{[(x-\xi)^2+\beta_\infty^2]^{3/2}}\,\mathrm{d}\xi \tag{11a}$$

where

$$H_a' \equiv h'/\beta_r^2 C_{p_r} \tag{11b}$$

$$m_a \equiv \frac{p_\infty M_\infty^2}{p_j M_r^2}\,\frac{\beta_r^2}{\beta_\infty^2} \tag{11c}$$

$$h_b' = \int_0^\infty \frac{S_b(\xi)\,\mathrm{d}\xi}{[(x-\xi)^2+\beta_\infty^2]^{3/2}} \tag{11d}$$

The function $\mathcal{P}_a$ is the axisymmetric equivalent of jet exit plane function discussed for Eq. (7), the a_n's are zero of Bessel functions of order one, J_1, and S is the source distribution related to the interface or boattail shape through Eq. (11a) or (11d). Equations (10) and (11) are to be solved in a coupled mode to define the interface shape. It is noted, however, that Eq. (11a) is Fredholm's integral equation of the first kind and is an extremely ill-posed problem to be solved directly.[16,17] Since the aim of the present study is to demonstrate the basic technique formulated here, an approximate solution to Eq. (11a) was sought. Such an approximate solution was generated by assuming that the source distribution S is directly proportional to the local slope h' (a result from slender body theory). The constant of proportionality was set by matching the slope at the location where it had its maximum value. This produced a remarkably accurate solution, except in a small region near the exit plane. This error in the source distribution was accepted here and, as will be shown later, apparently only has a minor effect on the solutions obtained. Once the interface shape is solved for via the solution of Eqs. (10) and (11), the streamwise perturbation velocity in the supersonic region can be expressed as

$$\phi_x = u = -\phi_x(0,h_j) + \int_0^x \frac{2}{\beta_r^2}h'(\xi)\,\mathrm{d}\xi + \sum_{n=1}^{\infty}\int_0^x \frac{2}{\beta_r^2}h'(\xi)\frac{J_0(a_n h)}{J_0(a_n)}\cos\left\{\frac{a_n(x-\xi)}{\beta_r}\right\}\mathrm{d}\xi \tag{12a}$$

and in the subsonic region

$$\phi_x = u = \int_0^\infty \frac{(x-\xi)S(\xi)}{[(x-\xi)^2+\beta_\infty^2]^{3/2}}\,\mathrm{d}\xi \tag{12b}$$

Similarly, other quantities, like transverse velocity and streamline curvature, can be evaluated.

Results and Discussion

A. Inviscid Flow Solutions

In order to demonstrate some fundamental aspects of the inviscid jet flow model approach used here, attention was initially focused on the two-dimensional case. To this end the normalized inviscid plume shape predicted through use of Eq. (7) for a two-dimensional coflowing jet with a uniform exit Mach number of $M_j = 1.9$ and a parallel freestream of $M_\infty = 0.8$ is shown in Fig. 3. These results correspond to a jet parameter value $m = 0.152$§ and, of course, hold for any level

§Consistent with a linear theory approach, in the definition of m, p_j/p_∞ is taken to be unity.

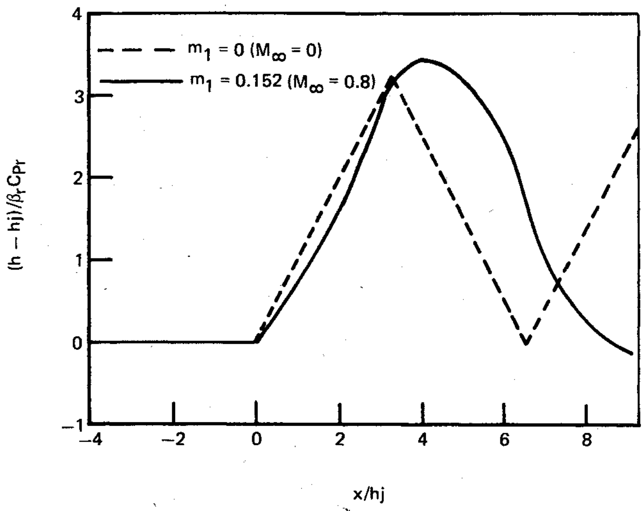
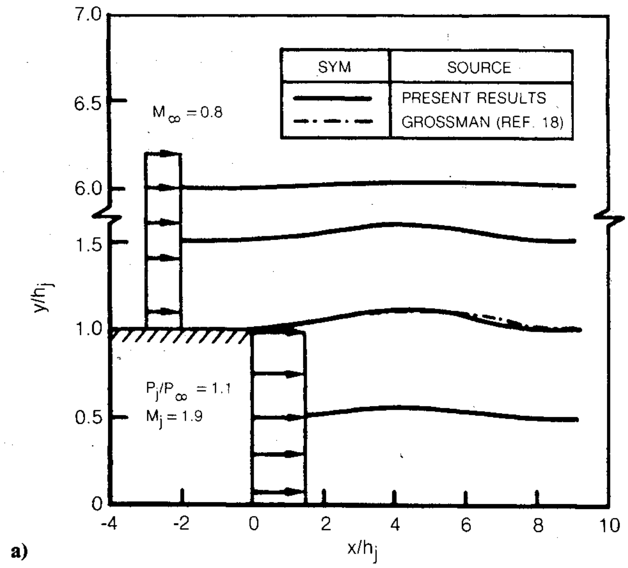


Fig. 3 Normalized inviscid streamline shape.



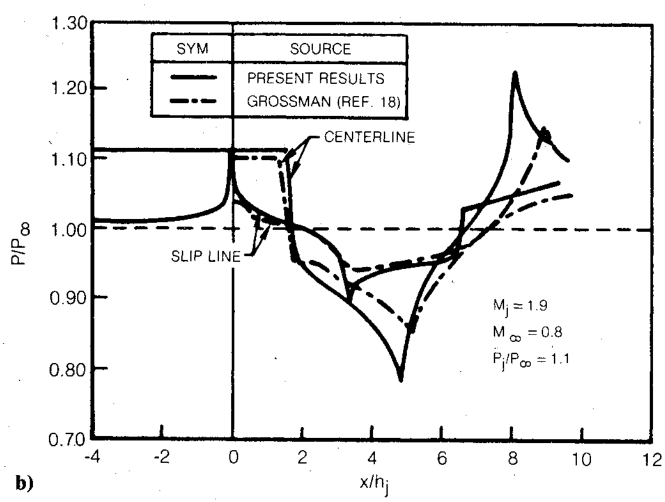


Fig. 4 Inviscid results for two-dimensional jets: a) plume shape; b) pressure distributions.

of the jet strength through Eq. (8a). Also shown is the freejet solution ($M_\infty = 0$, $m = 0$) yielding the classical diamond shape and giving a clear indication of the mainstream's influence on the jet structure. From this figure, it is quite clear that the coflowing stream has a significant impact on the plume shape both in terms of its basic shape and wavelength. In particular, it is seen that the mainstream flow increases the jet cell length and definitely removes the point expansion fan at the exit plane as manifested in the smooth (but rapid) initial growth of the plume. This effect is also evident in the attendant inviscid streamlines shown in Fig. 4a for the case of $M_\infty = 0.8$ and a jet pressure ratio $p_j/p_\infty = 1.1$. Note that the subsonic streamlines correctly show a small amount of upstream influence ahead of the jet exit and a progressive flattening as one moves laterally away from the jet plume centerline. Also shown in this figure is a favorable comparison with the slip line shape prediction of Grossman.[18] Figure 4b compares the slip line and centerline pressure predictions of Grossman[18] with present results. Grossman's solutions were obtained as numerical solutions of the complete set of inviscid flow governing equations and as such provide the most thorough assessment of the present exact solution of the approximate equations. Overall, the comparisons are excellent with the slight differences encountered being traceable to slight variations in the respective approximation procedures. As an example, it is noted that the slip line pressure levels virtually follow each other, except that sharper variations in gradient are observed in the present results. This is due to the fact that the current approach gives an analytically exact solution to the approximate model in these regions, while Grossman's approach provides a numerical approximation (and thus smoothing) to an analytically more exact model. The resulting lower gradient and slightly lower pressure level of the slip line pressure at the exit plane thereafter propagates throughout the flowfield causing the slight mismatch of the centerline pressure level extremums. Nonetheless, the overall comparisons are considered quite good and taken as an endorsement of the current approach for the class of slightly under- or overexpanded jet flows.

It is noted from the slip line pressure level given in this figure that there is a significant amount of upstream propagation in the subsonic flow, ahead of the jet exit plane, as it slows down to meet the exit pressure constraint. It would be anticipated that any boundary layer on the external surface might separate under such a large pressure gradient. As such, the pressure at $x = 0$ would be expected to decrease to relieve this strong gradient and the plume would undergo some amount of turning to meet this new pressure level. Inclusion of such separation effects is beyond the scope of the present study. Nonetheless, this issue is an important one and will be discussed further in review of the axisymmetric cases presented below.

In the current study, principal attention was focused on the case of axisymmetric coflowing jets as represented in Eq. (10). Solutions were obtained for essentially the same conditions as the two-dimensional case described above (i.e., $M_j = 1.9$, $M_\infty = 0.8$, and pressure ratios of $p_j/p_\infty = 1.1$ and 1.4). The plume shape obtained for the first of these conditions is shown in Fig. 5a along with the results of Salas[19] and Kardas and Presz.[20,21] Kardas and Presz[20] employed a method of characteristics solution for the supersonic jet flow as coupled to an iterative solution of the transonic small-distribution equation.[20] Upstream boundary-layer and separation effects are included in this model and thus it is not surprising to see differences in the plume shape near the jet exit plane. The overall agreement of the three solutions is taken as a strong endorsement of the current linear theory model for this range of test parameters. This ability of the current method to represent streamline shapes is of fundamental importance to the viscous flow solution scheme described above, since this is the critical information required by that approach to complete a solution of the jet mixing. Thus it is worthwhile to review some detailed aspects of the inviscid solutions since it will be found later to provide the fundamental structure of the entire plume flow. Figure 5b gives comparisons of the current inviscid centerline pressure distributions with those of Salas[17] and Kardas and Presz.[20] The general qualitative agreement

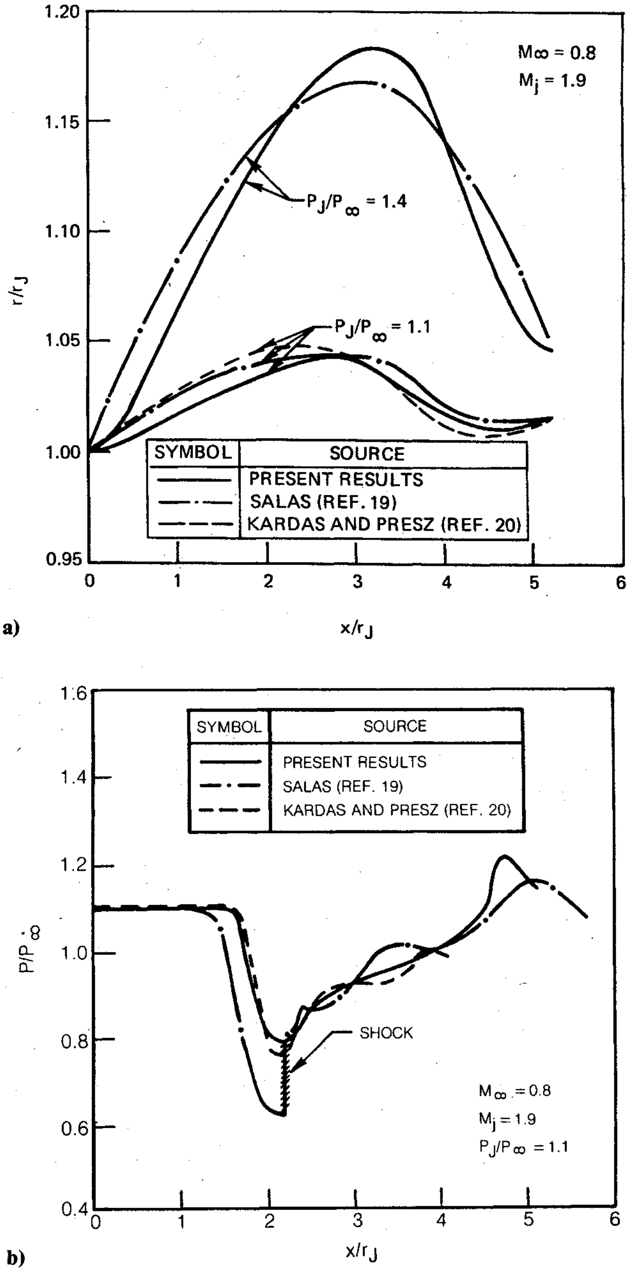


Fig. 5 Inviscid results for axisymmetric jets: a) plume shape; b) centerline pressure distribution.

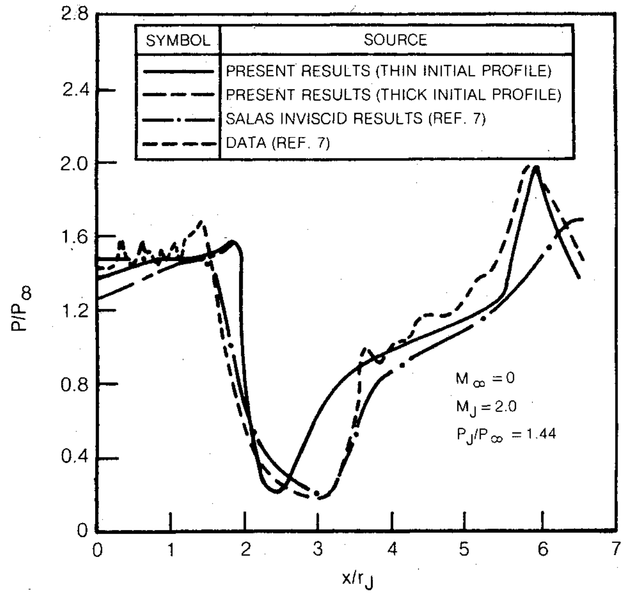


Fig. 6 Viscous pressure distributions for axisymmetric freejet.

with Salas' results[19] is apparently all that can be expected at low-pressure ratios due to the approximate nature of his slip line pressure model. It was found in Ref. 6 that the comparisons improve for higher pressure ratios, since Salas' subsonic model is more applicable to a large plume case.

The comparisons presented with the results of Kardas and Presz[20] were obtained in a slightly different fashion. As mentioned above, their method[21] includes upstream separation effects on the jet outer surface, producing more of a point turn at the exit (see Fig. 5a) than that given by the current inviscid theory. To allow a more direct assessment of the current linear theory model, Kardas and Presz[20] very obligingly reran their code using the current slip line shapes of Fig. 5a for $p_j/p_\infty = 1.1$ yielding the excellent comparison of Fig. 5b. This step provides a direct assessment of the linear theory model itself and proves that it is capable of accurately predicting the inviscid flow for jet parameter ranges of current interest. As a secondary issue, it was also found that the plume flow characteristics are critically dependent on the slip line shape at the exit plane. In particular, it was also found that the pressure rise to the exit plane along the outer surface of the jet strongly influences the interior jet flow. This important point and its implications are discussed further in Ref. 6.

B. Viscous Flow Results

Initial interest was focused on an underexpanded freejet test case since experimental data were available for detailed comparisons. Solutions were obtained here for the experimental conditions considered by Seiner[7]—that of a uniform cold jet at $M_j = 2.0$ and a pressure ratio of $p_j/p_\infty = 1.44$. Figure 6 gives a comparison of the present results and those presented in Ref. 7 which also included the inviscid results of Salas[19] as shown. In the current study, the viscous solutions were obtained using both thin and thick jet exit plane boundary layers to ascertain this influence on the centerline pressure signatures. As can be seen in Fig. 6, this influence is only felt near the exit plane and is believed to be nonphysical and only a manifestation of numerical approximations. This occurs due to the use of inviscid streamline curvatures (i.e., zero shear layer thickness) to integrate the normal momentum equation across the shear layer at the exit plane, where viscous effects would be expected to cause a local modification that would provide the correct pressure rise from the freestream level up to the jet exit condition. Aside from this slight discrepancy, the current solution gives extremely good comparison with the experimental pressure levels and only misses slightly in predicting the position of the minimum and maximum points. Apparently the weak shockwave-like structure that occurs near $x/r_j = 3$ is "captured" and smeared out by the current approach without numerical incident.

Attention was focused next on assessing the viscous effects in a coflowing stream for underexpanded jets. Robinson et al.[8] have presented total pressure and temperature measurements for an underexpanded ($p_j/p_\infty = 1.147$) supersonic ($M_j = 1.183$) hot ($T_{0j}/T_{0\infty} = 2.658$) jet coflowing in a subsonic ($M_\infty = 0.6$) stream. A slight change had to be made in the turbulence model for this case. The turbulence model, as described earlier, keys on the velocity defect between the freestream and jet core flow to estimate the shear layer thickness of the jet. For the current case, the experimental data showed a large variation in total temperature across the exit plane, which with the assumption of uniform exit Mach number, translates to a highly nonuniform jet velocity profile. Thus identification of a jet core velocity became difficult and an alternate strategy was established.

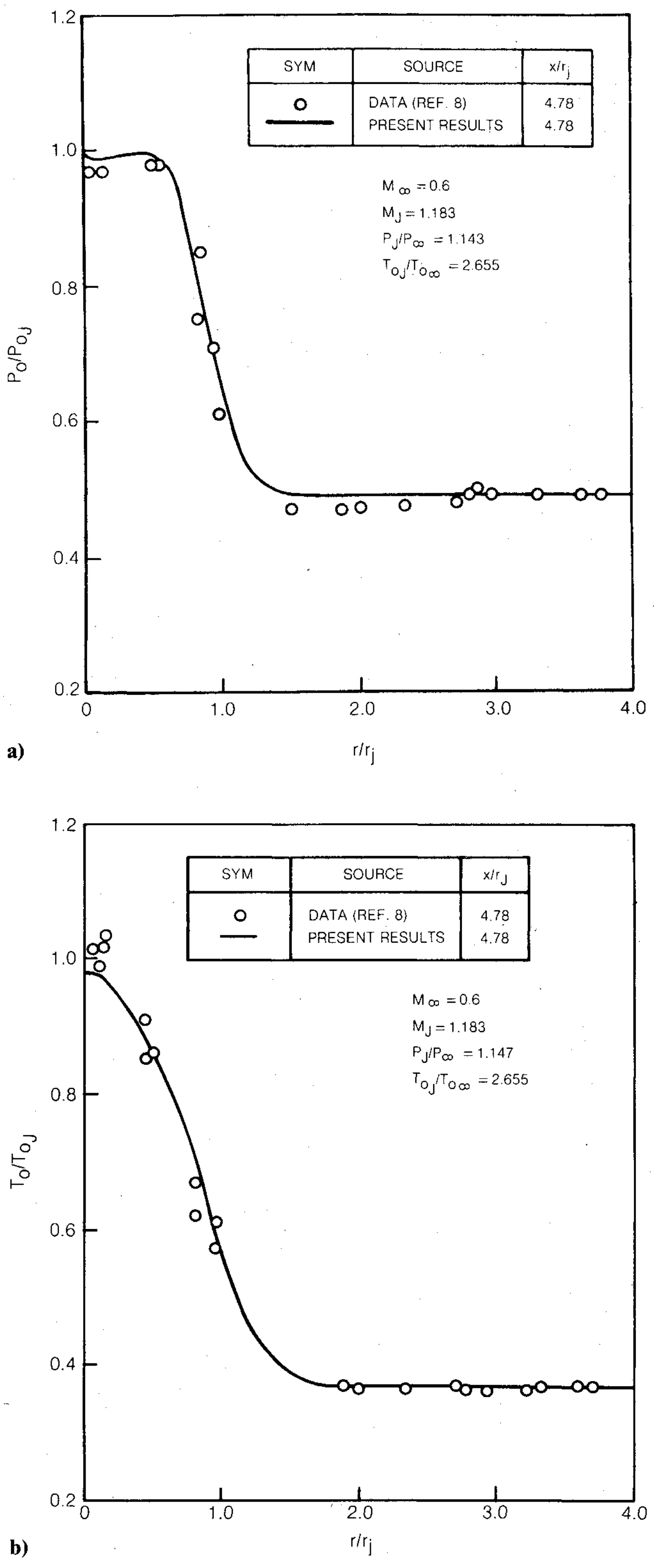


Fig. 7 Viscous results for axisymmetric coflowing jets: a) stagnation pressure distributions; b) stagnation temperature distributions.

For this case, the jet core region, and thus the shear layer thickness was established based on the total pressure defect behavior. Thus the transition to the fully turbulent jet mixing model was delayed until the centerline total pressure dropped 1% below its jet exit plane level.

The resulting solutions, using experimental data to estimate the jet exit plane conditions, are shown in Figs. 7a and 7b along with the experimental data at approximately five jet radii aft of the nozzle exit. As can be seen from these figures, the computed results compare very well with the data, thus verifying the overall approach and turbulence model employed herein.

Concluding Remarks

The detailed comparisons given here verify the proposed approach for calculating slightly underexpanded coflowing jet flow characteristics. The parabolic marching method for representing viscous mixing was found to be accurate, stable, and efficient. The inviscid solution scheme needed to provide approximate streamline curvatures was found to provide an accurate and simple representation of inviscid supersonic underexpanded jets in coflowing subsonic streams. Future work should be directed toward obtaining detailed experimental data for the critical assessment of the current model and extension of the concept to the three-dimensional level.

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